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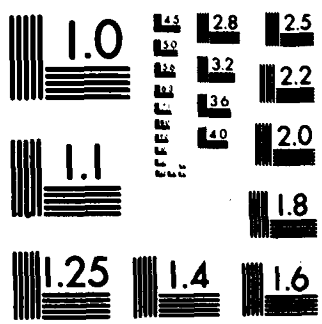

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A PROOF OF EXISTENCE OF A UNIQUE SOLUTION TO THE PROBLEM OF ADAPTIVE BEAMFORMING SUBJECT TO LINEAR CONSTRAINTS

by

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SUMMARY

A description of adaptive beamforming in the presence of linear constraints is given. It is proved that if the number of constraints does not exceed a stated maximum then the adaptive process determines one unique beam pattern. The proof is first written out explicitly using elementary algebra, and second in matrix form. The matrix form has the advantage of compactness and makes possible the writing of a simple formula for the weights which themselves define the beam pattern.

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1 INTRODUCTION

Adaptive beamforming (ABF) is a method of signal processing, explained in section 3 below. It is receiving considerable attention at the moment with a view to its possible applications to various types of sensor arrays and especially passive directional sonar. Over recent years a quite voluminous literature has accumulated concentrated upon the numerical techniques needed to render the process practicable. However, none of the papers seen by this writer has addressed the question of the relationship between the constraints and the uniqueness of the solution, or of the number of constraints to which it may be made subject. It appears to be worthwhile to make good any such gap, and this Memorandum does so, for the case of linear constraints, by the use of elementary algebra only. The proof is written out explicitly in section 4, followed by its briefer translation into matrix language in section 5 where the formal solution is also given. Before doing this the basic formulae of the subject will be derived and an explanation of ABF provided.

2 BASIC FORMULAE

Consider an array of n sensors, the complex output (typically the Fourier transform of the time series derived by periodically sampling an actual signal) of the kth sensor being x_k, $k = 1, 2, \ldots, n$. In the associated signal processing the complex conjugates of the x_k, denoted by x_k^*, are multiplied by complex weights w_k appropriate to a particular 'look' direction (or beam), and summed to give the output

$$y = \sum_{k=1}^{n} x_k^* w_k \tag{1}$$

corresponding to which the output power in the beam which has been formed is

$$|y|^2 = yy^* = \sum_{k=1}^{n} x_k^* w_k \sum_{\ell=1}^{n} x_\ell w_\ell^*$$

$$= \sum_{k,\ell=1}^{n} x_k^* x_\ell w_k w_\ell^* \; . \tag{2}$$

What the processer actually delivers as an output for display is an average of the power over some period of time. Such averages are statistical expectations,

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denoted for convenience by E , so that

$$E(|y|^2) = \sum_{k,\ell=1}^{n} E(x_k^* x_\ell) w_k w_\ell^* \tag{3}$$

where
$$E(x_k^* x_\ell) = \iint x_k^* x_\ell\, p(x_k^*, x_\ell)\, dx_k^* dx_\ell \tag{4}$$

$p(x_k^*, x_\ell)$ being the probability density function, and the integrations being taken over all values of x_k^*, x_ℓ for all directions. Then, taking the complex conjugate of (4),

$$\left[E(x_k^* x_\ell)\right]^* = \iint x_k x_\ell^*\, p(x_k^*, x_\ell)\, dx_k dx_\ell^* \tag{5}$$

where p , which is just a real number between 0 and 1, remains unaltered during conjugation. But the probability distributions of x_k and x_k^* must be identical (if x_k is specified so is its conjugate x_k^* , and conversely), *ie* $p(x_k^*, x_\ell) = p(x_k, x_\ell^*)$, which substituted into (5) gives

$$\left[E(x_k^* x_\ell)\right]^* = \iint x_k x_\ell^*\, p(x_k, x_\ell^*)\, dx_k dx_\ell^* \quad ,$$

which by (4)

$$= E(x_k x_\ell^*) = E(x_\ell^* x_k) \quad .$$

Hence, writing for brevity $e_{k\ell} \equiv E(x_k^* x_\ell)$, it follows that

$$e_{\ell k} = e_{k\ell}^* \quad , \tag{6}$$

and substitution into (3) gives for the expected power

$$E(|y|^2) = \sum_{k,\ell=1}^{n} e_{k\ell} w_k w_\ell^* \quad . \tag{7}$$

3 THE PRINCIPLE OF ADAPTIVE BEAMFORMING

As usually formulated the objective of ABF is that of minimising the power given by the expression (7) above while maintaining unit response in one specified direction. This last requirement may be stated by means of a linear relationship between the w_k as can be seen by the following simple example:- suppose that, for an incident wave of given direction and of frequency f, a time delay t_k occurs between the first and kth sensors for corresponding points in the wave front, then apart from a proportionality factor (defining the amplitude of the wave and its phase at the first sensor) $x_k = e^{2\pi i f t_k}$. Substitution of this in (1) shows that, for unit response in the given direction, the w_k must satisfy

$$\sum_{k=1}^{n} e^{-2\pi i f t_k} w_k = 1 \tag{8}$$

which is the linear relationship referred to. For conventional beamforming (CBF) $w_k = T(k)e^{2\pi i f t_k}$ where $T(k)$, called the taper function, satisfies $\sum_{k=1}^{n} T(k) = 1$ consistently with (8).

The freedom of choice in $T(k)$ makes possible the variety of beam patterns available to CBF. There are serious limitations however. Thus, taking for example the case of a linear array, if diffraction secondaries (repeats of the main beam) are to be excluded from look directions within $\pm 90^\circ$ of broadside the hydrophone spacing must be less than a wavelength. Hence, for a restricted number of hydrophones (and in practice such restriction is clearly unavoidable), the array length will be correspondingly limited. Under these circumstances any attempt to improve sensitivity by narrowing the main beam results in increased side lobes - and conversely. In practice there always will be side lobes, and the further the main beam is tilted away from broadside the wider it is, and the larger some of the side lobes become. If then, while an attempt were being made to detect a signal of interest in a certain direction by steering the main beam there, a sufficiently powerful noise was being picked up by such a side lobe, the whole operation of the array could be confused. ABF deals with this problem by fixing the response of the array in the direction of interest, using a linear constraint on the weighting factors w_k (as illustrated by (8) above), and then exploiting the remaining available variability of the w_k

in re-shaping the beam pattern so as to minimize the total energy received from all directions.

4 EXTENSION TO MULTIPLE CONSTRAINTS AND PROOF THAT ABF HAS A UNIQUE SOLUTION

A generalization of the above ABF method to cover multiple constraints will now be considered. It will be proved that any number $m < n - 1$ (where n is the number of independent sensors) of arbitrary independent linear constraints may be imposed on the weights w_k , $k = 1, 2, \ldots, n,$ subject to which there exists a unique set of values of the w_k (given explicitly by formula (28) in section 5) for which the power given by expression (7) attains a minimum.

Let there be m postulated constraints which define the array response for m specified directions such that

$$\sum_{j=1}^{n} c_{rj}^{*} w_j = q_r \qquad (9)$$

where $r = 1, 2, \ldots, m;$ q_r being the prescribed response in the rth direction, the direction being determined by the coefficient c_{rj} . So far as the present argument is concerned the c_{ij} and q_r are quite arbitrary, and do not need to have the merely typical physical interpretations just given (vide section 6). Clearly $m < n - 1$ since there are only n available w_js in (9), and if $m = n$ they would be completely determined by equations (9) alone leaving no variability for attaining a minimum of the expression (7). In practice m would probably be much smaller than n to provide as much freedom as possible for variation of the w_j in making the minimum of $E(|y|^2)$ as small as possible.

The complex conjugates of equations (9), namely

$$\sum_{j=1}^{n} c_{rj} w_j^{*} = q_r^{*} \quad (r = 1,2,\ldots,m < n - 1) \qquad (10)$$

may also be written.

The m constraint equations (9) leave $n - m$ of the w_j free to be varied in search of minima of $E(|y|^2)$. This observation suggests that a way to find a minimum would be to use the equations (9) to eliminate m of the w_j from the formula (7) leaving $E(|y|^2)$ expressed in terms of the remaining $n - m$

variables. If each of these remaining w_j s were then arbitrarily varied and the consequential first-order variations in $E(|y|^2)$ set to zero there would result $n - m$ linear equations to determine the values of the $n - m$ w_j s. These determined values when substituted into equations (9) would provide m equations to give the values of the m w_j s which were originally eliminated. This discussion incidentally makes an argument for the existence of a unique set of w_j giving a stationary value (not necessarily a maximum or a minimum, the existence of which calls for further enquiry) of $E(|y|^2)$. When translated into the actual algebraic steps implied however the formulation becomes unattractively complicated. Also, in practice, no guidance could be given as to which of the w_j should first be eliminated, and if an unfortunate choice were made then badly conditioned equations might appear for the remaining w_j leading to numerical problems.

These difficulties may be circumvented and complete symmetry maintained throughout by treating all the w_j on an equal footing from the start. Then setting to zero the first-order changes in $E(|y|^2)$ consequent upon arbitrary variations in all the w_j would give n equations, while the constraints (9) give m equations, thus making in all $m + n$ equations between the n variables w_j. To preserve the required freedom of action, m extra variables, written $\lambda_r (r = 1,2,\ldots,m)$ are introduced and the following n equations written

$$\sum_{i=1}^{n} e_{ij} w_i - \sum_{r=1}^{m} c_{rj} \lambda_r = 0 \qquad (11)$$

for $j = 1, 2, \ldots, n$; with which, by taking the complex conjugates, the n equations

$$\sum_{i=1}^{n} e_{ij}^* w_i^* - \sum_{r=1}^{m} c_{rj}^* \lambda_r^* = 0 \qquad (12)$$

for $j = 1, 2, \ldots, n$ are associated.

The $m + n$ equations (9) and (11) together determine a unique set of values for the $w_i (i = 1,2,\ldots,n)$ and the $\lambda_r (r = 1,2,\ldots,m)$: the actual formulae for these unknowns, (27) and (28), are derived in section 5 where it is seen that the values of w_i do not depend in any way upon the λ_r which disappear from the argument as soon as they have served their purpose of maintaining

its symmetry. Henceforth in this section the w_i will be assumed to stand for this particular set of values which it will be proved give a minimum for $E(|y|^2)$. To this end vary these w_i by arbitrary amounts h_i, subject only to the constraints (9), to become $w_i + h_i (i = 1,2,\ldots,n)$. Let $E(|y|^2)_w$ stand for the value of $E(|y|^2)$ (defined by equation (7)) when expressed in terms of the w_i, and $E(|y|^2)_{w+h}$ its value when the w_i are replaced by $w_i + h_i$. Then the variation in $E(|y|^2)$ is, by use of (7)

$$
\begin{aligned}
\delta E(|y|^2) \equiv\ & E(|y|^2)_{w+h} - E(|y|^2)_w \\
=\ & e_{11}(w_1h_1^* + w_1^*h_1) + e_{12}(w_1h_2^* + w_2^*h_1) + \ldots + e_{1n}(w_1h_n^* + w_n^*h_1) \\
& + e_{21}(w_2h_1^* + w_1^*h_2) + e_{22}(w_2h_2^* + w_2^*h_2) + \ldots + e_{2n}(w_2h_n^* + w_n^*h_2) \\
& + \ldots \\
& + e_{n1}(w_nh_1^* + w_1^*h_n) + e_{n2}(w_nh_2^* + w_2^*h_n) + \ldots + e_{nn}(w_nh_n^* + w_n^*h_n) \\
& + \sum_{i,j=1}^{n} e_{ij}h_ih_j^* \quad .
\end{aligned}
\tag{13}
$$

The right hand side of (13) expresses $\delta E(|y|^2)$ as a sum of the first degree terms in h_i and h_i^* written in extenso, and of the second degree terms $h_ih_j^*$ under the summation sign. Concentrating to begin with upon the first degree terms, and collecting together separately all those in $h_i (i = 1,2,\ldots,n)$ and those in $h_i^* (i = 1,2,\ldots,n)$ they amount to

$$
\begin{aligned}
& \left(\sum_{i=1}^{n} e_{i1}w_i\right)h_1^* + \left(\sum_{i=1}^{n} e_{i2}w_i\right)h_2^* + \ldots + \left(\sum_{i=1}^{n} e_{in}w_i\right)h_n^* \\
& + \left(\sum_{j=1}^{n} e_{ij}w_j^*\right)h_1 + \left(\sum_{j=1}^{n} e_{2j}w_j^*\right)h_2 + \ldots + \left(\sum_{j=1}^{n} e_{nj}w_j^*\right)h_n
\end{aligned}
$$

which by use of (6) is equal to

$$\left(\sum_{i=1}^{n} e_{i1}w_i\right)h_1^* + \left(\sum_{i=1}^{n} e_{i2}w_i\right)h_2^* + \ldots + \left(\sum_{i=1}^{n} e_{in}w_i\right)h_n^*$$

$$+\left(\sum_{j=1}^{n} e_{j1}^*w_j^*\right)h_1 + \left(\sum_{j=1}^{n} e_{j2}^*w_j^*\right)h_2 + \ldots + \left(\sum_{j=1}^{n} e_{jn}^*w_j^*\right)h_n \quad .$$

By substitution from (11) and (12) the last expression can be written

$$\left(\sum_{r=1}^{m} c_{r1}\lambda_r\right)h_1^* + \left(\sum_{r=1}^{m} c_{r2}\lambda_r\right)h_2^* + \ldots + \left(\sum_{r=1}^{m} c_{rn}\lambda_r\right)h_n^*$$

$$+\left(\sum_{r=1}^{m} c_{r1}^*\lambda_r^*\right)h_1 + \left(\sum_{r=1}^{m} c_{r2}^*\lambda_r^*\right)h_2 + \ldots + \left(\sum_{r=1}^{m} c_{rn}^*\lambda_r^*\right)h_n$$

which upon re-arrangement in terms of λ_r , λ_r^* becomes

$$\lambda_1 \sum_{j=1}^{n} c_{ij}h_j^* + \lambda_2 \sum_{j=1}^{n} c_{2j}h_j^* + \ldots + \lambda_m \sum_{j=1}^{n} c_{mj}h_j^*$$

$$+ \lambda_1^* \sum_{j=1}^{n} c_{ij}^*h_j + \lambda_2^* \sum_{j=1}^{n} c_{2j}^*h_j + \ldots + \lambda_m^* \sum_{j=1}^{n} c_{mj}^*h_j \quad . \qquad (14)$$

Both the w_j and the $w_j + h_j$ are required to satisfy the constraints (9). Substitution then of each of these two sets of variables into each of the m equations (9) followed by subtraction leaves the m results

$$\sum_{j=1}^{n} c_{rj}^*h_j = 0$$

for $r = 1, 2, \ldots, m$. Similarly the w_j^* and $w_j^* + h_j^*$ satisfy (10), and accordingly

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$$\sum_{j=1}^{n} c_{rj} h_j^* = 0 \quad ,$$

also for $r = 1, 2, \ldots, m$. Thus, returning to the expression (14) it is seen that the coefficients of all the λ_r and λ_r^* are zero, and hence the expression vanishes identically - there is therefore no first order variation in $\delta E(|y|^2)$, and $E(|y|^2)$ has a stationary value at the unique set of values of $w_i (i = 1,2,\ldots,n)$ determined by (9) and (11).

The only remaining contribution to the variation in $E(|y|^2)$ caused by the h_i and h_i^* is the last term written on the right hand side of (13). This is an exact replica of the expression (7) for $E(|y|^2)$ with the w_k, w_ℓ^* replaced by h_i, h_j^* . But $|y|^2$ by its very contrivance is inherently and inevitably positive, and so also therefore must be its expectation $E(|y|^2)$. Thus the second order variation in the power is positive whatever the values of h_i (and their accompanying h_i^*) so long as the weights w_i satisfy the equations (11) and both the w_i and their varied values $w_i + h_i$ satisfy the constraints (9). In other words a unique minimum exists for the expected power for the chosen look direction, and this corresponds to a unique set of weights w_i with its associated beam pattern.

5 MATRIX FORMULATION OF THE PROOF

This section is devoted to a rendering of the rather lengthy exercise of section 4 into the much shorter matrix version. It incidentally makes possible the writing of a deceptively compact formula (28), for the optimum values of the weights sought by the ABF process.

As before the basic formulae derived in section 2 provide the starting point. The n complex outputs $x_k (k = 1,2,\ldots,n)$ are written as a column vector $\underline{x}$, and the n complex weights w_k similarly as $\underline{w}$. The letter T standing above and to the right of a matrix or vector denotes its complex transpose, *ie* its transpose with its elements replaced by their complex conjugates, also known as the Hermitian transpose. Thus the row vector of elements x_k^* is written $\underline{x}^T$, so that equation (1) appears at $y = \underline{x}^T \underline{w}$, and the expression (2) for the power as

$$|y|^2 = \underline{w}^T \underline{x}\underline{x}^T \underline{w} \quad . \tag{15}$$

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The product $\underline{x}\underline{x}^T$ of the column vector $\underline{x}$ with the row vector $\underline{x}^T$ appearing in (15) is a square $n \times n$ matrix of elements $x_k x_\ell^*$. Hence formula (3) for the expectation of the power may be written

$$E(|y|^2) = \underline{w}^T E(\underline{x}\underline{x}^T)\underline{w}$$

$$= \underline{w}^T R \underline{w} \quad , \tag{16}$$

where R is a square matrix of n rows and n columns of elements $E(x_k x_\ell^*) = e_{\ell k}$ according to the definition of $e_{k\ell}$ given toward the end of section 2. R is called the covariance matrix of the sensor outputs x_k.

From (6) it can be seen at once that

$$R^T = R \quad , \tag{17}$$

which is the defining property of a Hermitian matrix, and is essential to the argument.

The set of m constraints (9) takes the form

$$C^T \underline{w} = \underline{q} \tag{18}$$

where $C \equiv [c_{jr}]$ $(j = 1,2,\ldots,n \; : \; r = 1,2,\ldots,m)$ is a matrix of n rows and m columns, and $\underline{q} \equiv (q_r)$ is a column vector of the m stipulated responses.

The matrix equation

$$R\underline{w} - C\underline{\lambda} = 0 \quad , \tag{19}$$

involving the extra column vector $\underline{\lambda} = (\lambda_r)$ of unknowns $\lambda_r (r = 1,2,\ldots,m)$ as well as that of the w_k, is then written. Equations (18) and (19) are a pair of simultaneous equations in the vectors $\underline{w}$ and $\underline{\lambda}$ which determine both of them, as will be demonstrated shortly. Before doing this, however, it will be shown that the $\underline{w}$ thus determined, the optimum $\underline{w}_0$ say, makes the $E(|y|^2)$ in (16) a minimum.

To this end let $\underline{w}_0$ be altered by a column vector $\underline{h} = (h_k)(k = 1,2,\ldots,n)$ which is arbitrary apart from $\underline{w}_0 + \underline{h}$ needing to satisfy the constraint (18). Then the consequential change in $E(|y|^2)$, found by substituting $\underline{w}_0 + h$ and $\underline{w}_0$ successively into the right hand side of (16) and subtracting, is

$$\delta E(|y|^2) = (\underline{w}_0 + \underline{h})^T R(\underline{w}_0 + \underline{h}) - \underline{w}_0^T R\underline{w}_0$$

$$= \underline{h}^T R\underline{w}_0 + \underline{w}_0^T R\underline{h} + \underline{h}^T R\underline{h} \quad . \tag{20}$$

But since $\underline{w}_0$ satisfies (19) then

$$R\underline{w}_0 = C\underline{\lambda} \tag{21}$$

with its Hermitian transpose

$$\underline{w}_0^T R = \underline{\lambda}^T C^T \quad , \tag{22}$$

remembering $R^T = R$ from (17).

Substituting (21) into the first term and (22) into the second term on the right of (20) gives

$$\delta E(|y|^2) = \underline{h}^T C\underline{\lambda} + \underline{\lambda}^T C^T \underline{h} + \underline{h}^T R\underline{h} \quad . \tag{23}$$

Again, both $\underline{w}_0 + \underline{h}$ and $\underline{w}_0$ satisfy the constraint (18), *ie*

$$C^T(\underline{w}_0 + \underline{h}) = C^T\underline{w}_0 + C^T\underline{h} = \underline{q}$$

and

$$C^T\underline{w}_0 = \underline{q} \quad , \tag{24}$$

whence by subtraction

$$C^T\underline{h} = 0 \tag{25}$$

of which the Hermitian transpose

$$\underline{h}^T C = 0 \tag{26}$$

must also of course be true.

Substitution from (25) and (26) into respectively the second and first terms on the right of (23) then leaves

$$\delta E(|y|^2) = \underline{h}^T R\underline{h}$$

only, which is just the expression on the right of (16) for $E(|y|^2)$ with $\underline{w}$ replaced by $\underline{h}$. But $E(|y|^2)$ is positive whatever values its variables may take. Thus $\delta E(|y|^2)$ is positive and so, for $\underline{w}$ equal to the $\underline{w}_0$ determined by (21) and (24), $E|y|^2)$ is a minimum.

For completeness the actual solutions for $\underline{w}_0$ and $\underline{\lambda}$ will now be derived. Substitution for $\underline{w}_0$ from (21) into (24), or from (19) into (18) gives

$$C^T R^{-1} C\underline{\lambda} = \underline{q} ,$$

whence

$$\underline{\lambda} = \frac{\underline{q}}{C^T R^{-1} C} , \tag{27}$$

and this substituted into (21) gives

$$\underline{w}_0 = \frac{R^{-1} C\underline{q}}{C^T R^{-1} C} . \tag{28}$$

The formula last written is the explicit solution of the array processing problem using ABF . It is seen, however, to require the evaluation of R^{-1} , the inverse of the covariance matrix whose order equals the number n of sensors in the array. Such inversions for large n are a notoriously difficult problem. Because of this, much of the effort in support of ABF has been aimed toward methods of seeking the minimum of $E(|y|^2)$ otherwise by finding the most rapid descent path across the surface of the function $E(|y|^2)$ in w space. This paper demonstrates that for a given set of covariances and defined constraints, such paths all lead to a unique minimum.

6 CONCLUDING REMARKS AND CHOICE OF CONSTRAINTS

This Memorandum gives a rigorous proof of the existence of a unique solution to the minimisation problem of ABF in the particular case of linear constraints but, also for this particular case, generalized as far as is possible to cover the maximum number of such constraints. It gives a formula, (28), for the solution in terms of the noise field as expressed by the covariance matrix R and in terms of the constraints through their directions represented by the matrix C and the vector of their magnitudes q .

As pointed out in the remarks following equation (9) in section 4, these represent but one possible choice of meanings for C and $\underline{q}$, and the mathematical argument is unaffected whatever physical interpretation they are given. Some examples of other meanings are discussed below.

The practical significance of the theorem is that whatever method be used to seek the minimum it can lead only to one unique solution. Thus, for example, if a procedure of successive approximations were adopted there would be no danger of convergence toward some spurious local minimum, as there exists none but just the one true minimum to be sought.

On the other hand, the analysis of this paper has nothing to say about the beamshape produced by the particular ABF process studied. It is consideration of this, however, which can strongly influence the choice of constraints in the first place. After all, the whole purpose of ABF is to produce some sort of 'best possible' beam pattern. If for example the signal of interest should lie close to, but not exactly along, the direction demanded by a constraint, then the ABF process may inherently attempt to reject the genuine signal while concentrating upon the constrained direction, forming a very narrow beam in the process. One way to avoid this hazard would be by requiring the average response or responses over an assigned sector or sectors to take prescribed values, and this, leading as it does to linear constraints, is covered by the present work. The application of this approach is the subject of a study by G.J. Lawson of Radio and Navigation Department, RAE to be published in due course.

A quite different approach depends upon the fact that very narrow beams are generally associated with the occurrence of very large positive and negative values of the weights. The ABF process could then be prevented from generating such beams if a constraint of the form limiting the sum of the squares of the weights to a reasonable value were imposed. Such quadratic constraints have been studied by J. Hudson of Loughborough University of Technology - they fall outside the scope of this Memorandum.

If yet another method of avoiding over-narrow beams were used, namely requiring that the rate of change of response with look direction be limited, then although the constraints are still linear they then take the form of inequalities. Again the proof given above does not apply. The same would be true if, in order to maintain a prescribed beam width, constraints were imposed which require the responses to exceed a defined value in certain directions close to the look direction.

In short, this paper establishes the basic uniqueness theorem in the most general form for ABF with linear constraints, but makes no contribution to the theory when the constraints are quadratic or are expressed as inequalities.

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